The Lagrangian Approach to Fluid Dynamics

A Primer on Variational Methods for Continuous Media

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July 1, 2026

Abstract

The Lagrangian framework, introduced in the companion paper *The Lagrangian Approach to Physics*, was originally developed for systems of discrete particles. Extending it to fluids—continuous media with infinitely many degrees of freedom—raises fundamental challenges that took two centuries to resolve. This paper traces that extension at an accessible level, from the basic fluid Lagrangian through the treatment of vorticity, rotating fluids, superfluids, and the modern geometric formulation. The goal is to show how the same variational principle that governs a pendulum also governs hurricanes, ocean currents, neutron stars, and superfluid helium.

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1 Introduction: Why Fluids Are Different

A pendulum has one degree of freedom. A double pendulum has two. A fluid filling a room has *infinitely many*: the velocity, density, and pressure at every point in space, evolving continuously in time.

This jump from finite to infinite degrees of freedom is not merely quantitative. It introduces qualitative difficulties:

- **Which variables?** In particle mechanics, the generalised coordinates $q_i(t)$ are obvious (positions, angles). In fluid mechanics, should we track where each fluid parcel goes (*Lagrangian* description), or watch what flows past each fixed point in space (*Eulerian* description)?
- **Constraints are pervasive.** Mass conservation, incompressibility, entropy conservation—these are not optional side conditions; they shape the very structure of the variational principle.
- **Vorticity is topological.** Rotation in a fluid is not a simple coordinate; vortex lines are conserved structures that resist naive variational treatment.

Despite these obstacles, the Lagrangian framework does extend to fluids—and when it does, the rewards are the same as for particles: coordinate freedom, automatic conservation laws via Noether's theorem, and a natural gateway to quantum and relativistic generalisations.

This paper builds on the companion primer *The Lagrangian Approach to Physics*, which covers the particle-mechanics foundations. We assume familiarity with the action principle and the Euler–Lagrange equations; all fluid-specific concepts are developed from scratch.

2 Two Descriptions of Fluid Motion

Before writing a Lagrangian, we must choose how to describe the fluid. There are two classical viewpoints, both named after the same historical figures who shaped the Lagrangian method itself.

2.1 The Lagrangian (Material) Description

Label each fluid parcel by its initial position $\mathbf{a}$ at time $t = 0$. The dynamical variable is the **flow map** $\mathbf{x}(\mathbf{a}, t)$: where parcel $\mathbf{a}$ ends up at time t.

This is the direct analogue of particle mechanics—each parcel is a "particle" with a trajectory $\mathbf{x}(t)$. The Lagrangian is straightforward:

$$L = \int \left[\frac{1}{2}\rho_0(\mathbf{a}) \left| \frac{\partial \mathbf{x}}{\partial t} \right|^2 - \rho_0(\mathbf{a})\, \epsilon\big(\rho(\mathbf{x}, t),\, s(\mathbf{a})\big) - \rho_0(\mathbf{a})\, \Phi(\mathbf{x}) \right] \mathrm{d}^3 a\,, \tag{1}$$

where $\rho_0(\mathbf{a})$ is the initial density, ϵ is the internal energy per unit mass, $s(\mathbf{a})$ is the entropy (carried with each parcel), and Φ is any external potential (gravity). The current density is determined by volume conservation: $\rho(\mathbf{x}, t) = \rho_0(\mathbf{a})/J$, where $J = \det(\partial\mathbf{x}/\partial\mathbf{a})$ is the Jacobian of the flow map.

Remark 2.1. In this description, the Euler–Lagrange equations yield Newton's second law for each fluid element, with pressure gradients emerging naturally from the internal energy term. No constraint forces are needed.

2.2 The Eulerian (Spatial) Description

Fix attention on a point $\mathbf{x}$ in space and record the velocity $\mathbf{v}(\mathbf{x}, t)$, density $\rho(\mathbf{x}, t)$, and pressure $p(\mathbf{x}, t)$ as fluid streams past.

This is how we usually write fluid equations (the Navier–Stokes equations, for instance), and it is far more convenient for computation. But it creates a problem for variational principles: the velocity field $\mathbf{v}(\mathbf{x}, t)$ cannot be varied freely, because independent variations of $\mathbf{v}$ at different points would break the connection between neighbouring fluid parcels.

Resolving this tension—writing a true variational principle in Eulerian variables—is the central technical challenge of fluid Lagrangian theory, and it occupied mathematicians from the 1850s to the 1990s.

3 The Basic Fluid Lagrangian

3.1 Kinetic Minus Potential—For a Continuum

The fluid Lagrangian has exactly the same philosophical structure as for particles:

$$\boxed{L = \int \left[\underbrace{\tfrac{1}{2}\rho\,|\mathbf{v}|^2}_{T\text{: kinetic energy}} - \underbrace{\rho\,\epsilon(\rho, s)}_{U\text{: internal energy}} - \underbrace{\rho\,\Phi}_{V\text{: external potential}} \right] \mathrm{d}^3x\,.} \tag{2}$$

Each term is a *density* (energy per unit volume), integrated over the entire fluid domain. The action is $S = \int_{t_1}^{t_2} L\,\mathrm{d}t$.

Term	**Symbol**	**Physical Meaning**
Kinetic	$\frac{1}{2}\rho\lvert\mathbf{v}\rvert^2$	Energy of bulk fluid motion
Internal	$\rho\,\epsilon(\rho, s)$	Thermodynamic energy (pressure, temperature); encodes the equation of state
External	$\rho\,\Phi$	Gravitational or other body-force potential

3.2 What the Internal Energy Encodes

The function $\epsilon(\rho, s)$ is the equation of state in disguise. The pressure follows from thermodynamics:

$$p = \rho^2 \left.\frac{\partial\epsilon}{\partial\rho}\right|_s . \tag{3}$$

Different choices of ϵ give different fluids:

- **Incompressible**: $\epsilon = \text{const}$ (pressure becomes a Lagrange multiplier enforcing $\nabla\cdot\mathbf{v} = 0$).
- **Ideal gas**: $\epsilon \propto \rho^{\gamma-1}$, giving $p = K\rho^\gamma$ (polytropic).
- **Shallow water**: $\epsilon = \frac{1}{2}gh$, where h is the water depth (used in oceanography).
- **Superfluid**: ϵ includes the quantum pressure $(\hbar^2/2m)\,|\nabla\sqrt{\rho}|^2/\rho$ (see section 7).

4 The Vorticity Problem and Its Solutions

4.1 The Problem

If you naively vary the Eulerian Lagrangian (eq. (2)) with respect to $\mathbf{v}$, treating $\mathbf{v}(\mathbf{x}, t)$ as an unconstrained field, you obtain only *irrotational* (curl-free) flow:

$$\mathbf{v} = \nabla\phi \quad \Longrightarrow \quad \boldsymbol{\omega} \equiv \nabla \times \mathbf{v} = 0.$$

This misses all vortical motion—tornadoes, vortex rings, turbulence, rotating storms—which is most of fluid dynamics.

The root cause is that the Eulerian velocity field has a hidden structure: not all $\mathbf{v}$ fields correspond to physically realisable flows. A legitimate velocity field must be the Eulerian image of a material flow map that preserves parcel identity. Ignoring this constraint loses vorticity.

4.2 Solution 1: Clebsch Variables (1859)

Alfred Clebsch (Clebsch, 1859) showed that *any* velocity field in three dimensions can be decomposed as

$$\mathbf{v} = \nabla\phi + \alpha\,\nabla\beta, \tag{4}$$

where ϕ, α, β are scalar functions called **Clebsch potentials**. The vorticity is then

$$\boldsymbol{\omega} = \nabla\alpha \times \nabla\beta,$$

which is generically non-zero. The Lagrangian is varied with respect to ϕ, α, β (and ρ, s) as independent fields. The Euler–Lagrange equations correctly reproduce the full Euler equations for an ideal fluid, including vortex dynamics.

Remark 4.1. The Clebsch representation is not unique and can develop singularities at vortex cores. It works beautifully for smooth flows but requires care for concentrated vortices.

4.3 Solution 2: Lin Constraints (1963)

Lin (1963) proposed adding Lagrange multipliers that enforce *particle conservation*—the constraint that each fluid parcel retains its identity label $\mathbf{a}$ as it moves:

$$\frac{\partial a_i}{\partial t} + \mathbf{v} \cdot \nabla a_i = 0, \qquad i = 1, 2, 3. \tag{5}$$

The constrained Lagrangian becomes

$$L' = L + \int \lambda_i \left(\frac{\partial a_i}{\partial t} + \mathbf{v} \cdot \nabla a_i \right) \mathrm{d}^3 x\,,$$

where λ_i are multiplier fields. Varying $\mathbf{v}$ now includes the constraint forces that maintain parcel coherence, and the full vortical Euler equations emerge.

4.4 Solution 3: Euler–Poincaré Reduction (1998)

The modern approach, due to Holm et al. (1998), recognises the fluid as a dynamical system on the *group of diffeomorphisms*—the infinite-dimensional Lie group of smooth, invertible maps of space to itself. The Lagrangian is defined on the Lie algebra (velocity fields), and variations are constrained by the group structure:

$$\delta\mathbf{v} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{v}, \mathbf{w}], \tag{6}$$

where $\mathbf{w}$ is an arbitrary vector field vanishing at the endpoints, and $[\mathbf{v}, \mathbf{w}] = (\mathbf{v} \cdot \nabla)\mathbf{w} - (\mathbf{w} \cdot \nabla)\mathbf{v}$ is the Lie bracket (negative of the commutator of the flows).

This elegant formulation:

- Recovers the full Euler equations with vorticity,
- Explains Kelvin's circulation theorem (Thomson, 1869) as a Noether consequence of particle-relabeling symmetry,
- Extends naturally to magnetohydrodynamics, elasticity, complex fluids, and plasma physics.

5 Conservation Laws from Symmetry

Noether's theorem, applied to the fluid Lagrangian, yields the fundamental conservation laws of fluid dynamics automatically:

Symmetry	**Conserved Quantity**	**Fluid Expression**
Time translation	Energy	$E = \int \left[\frac{1}{2}\rho\lvert\mathbf{v}\rvert^2 + \rho\epsilon + \rho\Phi\right] \mathrm{d}^3x$
Space translation	Momentum	$\mathbf{P} = \int \rho\,\mathbf{v}\ \mathrm{d}^3x$
Rotation	Angular momentum	$\mathbf{L} = \int \rho\,(\mathbf{x}\times\mathbf{v})\ \mathrm{d}^3x$
Particle relabeling	Circulation	$\Gamma = \oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{l}$ (Kelvin's theorem)
Entropy label	Entropy per parcel	$Ds/Dt = 0$ (adiabatic flow)

5.1 Kelvin's Circulation Theorem

The most profound of these is the last classical entry: *particle-relabeling symmetry* (the physics doesn't depend on how we label the parcels) implies **Kelvin's circulation theorem**—the circulation Γ around any material loop is conserved:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{C(t)} \mathbf{v}\cdot\mathrm{d}\mathbf{l} = 0. \tag{7}$$

In the Newtonian approach, Kelvin's theorem must be *proved* from the Euler equations by a calculation. In the Lagrangian approach, it is an *automatic consequence* of a symmetry—a arguably deeper and more illuminating derivation.

5.2 Ertel's Potential Vorticity

A related Noether consequence is the conservation of **potential vorticity**:

$$q = \frac{\boldsymbol{\omega}\cdot\nabla s}{\rho}, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = 0, \tag{8}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the vorticity and s is the entropy. This quantity, discovered by Ertel (1942), is the single most important diagnostic in atmospheric and oceanic science. Its conservation follows from the combined particle-relabeling and entropy-label symmetries of the fluid Lagrangian (Shepherd, 1990).

6 Rotating Fluids

Much of geophysics—weather, ocean circulation, planetary interiors—involves fluids in rotating reference frames. The Lagrangian formulation handles this with a simple substitution.

6.1 The Rotating-Frame Lagrangian

If the frame rotates with angular velocity $\boldsymbol{\Omega}$, the inertial velocity of a parcel at position $\mathbf{x}$ with rotating-frame velocity $\mathbf{v}'$ is

$$\mathbf{v}_{\text{inertial}} = \mathbf{v}' + \boldsymbol{\Omega}\times\mathbf{x}. \tag{9}$$

Substituting into $T = \frac{1}{2}\rho|\mathbf{v}_{\text{inertial}}|^2$ and expanding:

$$\boxed{\mathcal{L}_{\text{rot}} = \underbrace{\tfrac{1}{2}\rho\,|\mathbf{v}'|^2}_{\text{kinetic}} + \underbrace{\rho\,\mathbf{v}'\cdot(\boldsymbol{\Omega}\times\mathbf{x})}_{\text{Coriolis coupling}} + \underbrace{\tfrac{1}{2}\rho\,|\boldsymbol{\Omega}\times\mathbf{x}|^2}_{\text{centrifugal}} - \rho\,\epsilon - \rho\,\Phi.} \tag{10}$$

6.2 Physics of Each Term

1. **Kinetic term** ($\frac{1}{2}\rho|\mathbf{v}'|^2$): The standard kinetic energy measured in the rotating frame.
2. **Coriolis coupling** ($\rho\,\mathbf{v}'\cdot(\boldsymbol{\Omega}\times\mathbf{x})$): This is *first order* in $\mathbf{v}'$, which is unusual—most Lagrangians have only quadratic velocity dependence. The Euler–Lagrange equations turn this into the **Coriolis force** $-2\rho\,\boldsymbol{\Omega}\times\mathbf{v}'$, the force responsible for the rotation of weather systems and ocean gyres.
   The first-order velocity dependence gives the equations a *gyroscopic* structure: the Coriolis force does no work (it is perpendicular to the velocity) but redirects motion, producing the characteristic spiral patterns of geophysical flows.
3. **Centrifugal term** ($\frac{1}{2}\rho|\boldsymbol{\Omega}\times\mathbf{x}|^2$): This is velocity-independent and acts as a modification to the effective potential. It is usually absorbed into a *geopotential* $\Phi_{\text{eff}} = \Phi - \frac{1}{2}|\boldsymbol{\Omega}\times\mathbf{x}|^2$, which accounts for why the Earth is an oblate spheroid (the equator bulges outward to balance centrifugal and gravitational forces).

6.3 The Rossby Number and Geostrophic Balance

When rotation dominates (small Rossby number $\text{Ro} = U/(fL) \ll 1$, where $f = 2\Omega\sin\phi$ is the Coriolis parameter), the Coriolis term in the Lagrangian dominates the kinetic term. The leading-order balance gives *geostrophic flow*:

$$f\,\hat{\mathbf{z}}\times\mathbf{v}' = -\frac{1}{\rho}\nabla p,$$

where flow is perpendicular to the pressure gradient—explaining why winds circulate *around* pressure centres rather than blowing directly from high to low pressure.

This result follows directly from the Euler–Lagrange equations of eq. (10) in the limit where the $\frac{1}{2}\rho|\mathbf{v}'|^2$ term is negligible compared to the Coriolis coupling.

7 Superfluids: Quantised Vorticity

Superfluids—liquid helium below 2.17 K, ultracold atomic gases, and (in theory) the neutron fluid inside neutron stars—exhibit a dramatically different kind of fluid dynamics, governed by quantum mechanics. The Lagrangian framework extends naturally to this domain.

7.1 The Gross–Pitaevskii Lagrangian

A superfluid at zero temperature is described by a macroscopic wavefunction $\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}\,e^{i\theta(\mathbf{x},t)}$. The Lagrangian density is

$$\mathcal{L}_{\text{GP}} = i\hbar\,\psi^*\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}|\nabla\psi|^2 - V_{\text{ext}}|\psi|^2 - \frac{g}{2}|\psi|^4, \tag{11}$$

where m is the atomic mass and g is the interaction strength.

The superfluid velocity is determined by the phase gradient:

$$\mathbf{v}_s = \frac{\hbar}{m}\nabla\theta. \tag{12}$$

7.2 Irrotational—Except at Vortex Cores

Since $\mathbf{v}_s = (\hbar/m)\nabla\theta$, the vorticity is identically zero everywhere:

$$\nabla \times \mathbf{v}_s = \frac{\hbar}{m}\nabla \times \nabla\theta = 0 \qquad \text{(wherever } \psi \neq 0\text{)}.$$

But θ can have *topological singularities*—points where $\psi = 0$ and the phase winds by $2\pi n$ around a closed loop. These are **quantised vortices**: lines of zero density around which the circulation is exactly

$$\Gamma = \oint \mathbf{v}_s \cdot \mathrm{d}\mathbf{l} = n\,\frac{h}{m}, \qquad n = 0, \pm 1, \pm 2, \ldots \tag{13}$$

Unlike classical vortices, which can have any strength and diffuse over time, superfluid vortices have *exactly integer* circulation and are topologically stable—they cannot be created or destroyed except by pair creation/annihilation or by hitting a boundary.

7.3 Rotation in a Superfluid

A classical fluid in a rotating bucket settles into solid-body rotation ($\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{x}$). A superfluid *cannot* do this, because solid-body rotation requires non-zero vorticity everywhere, which eq. (12) forbids.

Instead, the superfluid approximates solid-body rotation by nucleating an *array of quantised vortex lines*, each carrying one quantum of circulation h/m. The density of vortices per unit area is

$$n_v = \frac{2m\Omega}{h}, \tag{14}$$

the **Feynman–Onsager relation**. This was predicted theoretically by Feynman (1955) and Onsager (1949), and confirmed experimentally in rotating helium and, more recently, in Bose–Einstein condensates.

In the Lagrangian framework, this result emerges from minimising the energy functional in the rotating frame:

$$E_{\text{rot}} = E - \boldsymbol{\Omega} \cdot \mathbf{L},$$

where $\mathbf{L}$ is the angular momentum. Above a critical angular velocity Ω_c, the energy cost of creating a vortex is outweighed by the angular momentum gain, and vortices spontaneously nucleate.

8 The Two-Fluid Model

At finite temperature, superfluid helium (He II) behaves as if it were two interpenetrating fluids:

Component	**Carries**	**Velocity**
Superfluid (ρ_s)	No entropy, no viscosity	$\mathbf{v}_s$ (irrotational)
Normal fluid (ρ_n)	All entropy and viscosity	$\mathbf{v}_n$ (general)

with $\rho = \rho_s + \rho_n$.

The Lagrangian for this system, following Khalatnikov (2000), is:

$$\boxed{L = \int \left[\tfrac{1}{2}\rho_s|\mathbf{v}_s|^2 + \tfrac{1}{2}\rho_n|\mathbf{v}_n|^2 - \epsilon(\rho, s) - \rho\,\Phi\right] \mathrm{d}^3x\,,} \tag{15}$$

with the constraints $\nabla \times \mathbf{v}_s = 0$ (away from vortex cores) and $Ds/Dt = \partial_t s + \mathbf{v}_n \cdot \nabla s = 0$ (entropy is carried only by the normal fluid).

The Euler–Lagrange equations yield the celebrated **two-fluid equations** of Landau and Tisza, which predict remarkable phenomena:

- **Second sound**: Temperature waves (entropy oscillations carried by the normal fluid while the superfluid remains still)—a wave mode with no classical analogue.
- **The fountain effect**: Superfluid flows through narrow channels toward heat sources, producing jets of helium.
- **Mutual friction**: When quantised vortices are present, they scatter the normal fluid, coupling the two components.

9 Geophysical Fluid Dynamics: Salmon's Programme

Rick Salmon, in his influential works (Salmon, 1983, 1998), systematically applied the Lagrangian approach to the large-scale dynamics of the atmosphere and ocean. His key insight was that approximate models (quasi-geostrophic, shallow-water, primitive equations) should be derived by approximating the *Lagrangian first*, then taking variations—rather than approximating the equations of motion directly.

9.1 Why Approximate the Lagrangian?

If you start with exact equations and then make approximations, you risk breaking conservation laws. Kelvin's theorem, energy conservation, and potential vorticity conservation are all *consequences of symmetries in the Lagrangian* (Bretherton, 1970). If you approximate the Lagrangian instead, the resulting (approximate) equations automatically inherit properly modified conservation laws from the approximate Lagrangian's symmetries via Noether's theorem.

9.2 The Shallow-Water Lagrangian

The simplest geophysical model is the rotating shallow-water equations, which describe a single layer of fluid with free surface height $h(\mathbf{x}, t)$ on a rotating planet:

$$L = \int \left[\tfrac{1}{2}\rho h \, |\mathbf{v}|^2 + \rho h \, \mathbf{v} \cdot (\boldsymbol{\Omega} \times \mathbf{x}) - \tfrac{1}{2}\rho g h^2\right] \mathrm{d}^2 x \, . \tag{16}$$

The Euler–Lagrange equations give the rotating shallow-water equations, which describe tsunamis, Kelvin waves, Rossby waves (the "weather waves" that steer mid-latitude storm tracks), and the basic structure of ocean gyres—all from a single, compact Lagrangian.

10 The Modern Geometric View

The deepest formulation of fluid Lagrangian theory, developed by Arnold (1966) and extended by Holm et al. (1998) and others (Holm et al., 2009), views fluid dynamics as *geodesic motion on an infinite-dimensional Lie group.*

10.1 Arnold's Insight

Arnold showed that the Euler equations for an ideal incompressible fluid describe **geodesics on the group of volume-preserving diffeomorphisms** SDiff(M), with the metric given by the kinetic energy:

$$\langle \mathbf{v}, \mathbf{v} \rangle = \int \tfrac{1}{2}\rho \, |\mathbf{v}|^2 \ \mathrm{d}^3 x \, .$$

Just as a free particle on a curved surface follows a geodesic (shortest path), a fluid with no external forces follows a geodesic on the "space of all possible rearrangements of the fluid."

10.2 Why This Matters

This geometric perspective:

- Explains **why the Euler equations are so hard**: the diffeomorphism group has infinite dimension and non-trivial curvature, making geodesics (fluid flows) inherently prone to exponential divergence—i.e., *turbulence*.
- Provides a **unified framework** for deriving all ideal fluid models (Euler, MHD, elasticity, liquid crystals, superfluids) as geodesics or Euler–Poincaré equations on different Lie groups with different “advected quantities” (density, entropy, magnetic field, director field, . . .).
- Connects fluid dynamics to **gauge theory**: the particle-relabeling symmetry is an infinite-dimensional gauge symmetry, and vorticity is the associated “curvature.”

11 A Summary of Fluid Lagrangians

System	**Lagrangian (density)**	**Key Feature**
Ideal fluid (material)	$\frac{1}{2}\rho_0\lvert\dot{\mathbf{x}}\rvert^2 - \rho_0\epsilon(\rho_0/J, s)$	Direct particle analogy; Jacobian J tracks volume
Ideal fluid (Eulerian)	$\frac{1}{2}\rho\lvert\mathbf{v}\rvert^2 - \rho\epsilon(\rho, s)$	Requires Clebsch / Lin / Euler–Poincaré for vorticity
Rotating fluid	$\frac{1}{2}\rho\lvert\mathbf{v}'\rvert^2 + \rho\mathbf{v}'\cdot(\mathbf{\Omega}\times\mathbf{x}) + \frac{1}{2}\rho\lvert\mathbf{\Omega}\times\mathbf{x}\rvert^2 - \rho\epsilon$	Coriolis is 1st-order in $\mathbf{v}'$; centrifugal modifies potential
Shallow water (rotating)	$\frac{1}{2}\rho h\lvert\mathbf{v}\rvert^2 + \rho h\mathbf{v}\cdot(\mathbf{\Omega}\times\mathbf{x}) - \frac{1}{2}\rho g h^2$	Rossby waves, tsunamis, ocean gyres
Superfluid (Gross–Pitaevskii)	$i\hbar\psi^*\partial_t\psi - \frac{\hbar^2}{2m}\lvert\nabla\psi\rvert^2 - \frac{g}{2}\lvert\psi\rvert^4$	Velocity $= (\hbar/m)\nabla\theta$; quantised vortices
Two-fluid (Khalatnikov)	$\frac{1}{2}\rho_s\lvert\mathbf{v}_s\rvert^2 + \frac{1}{2}\rho_n\lvert\mathbf{v}_n\rvert^2 - \epsilon(\rho, s)$	Second sound; fountain effect; mutual friction

12 Summary and Perspective

1. The fluid Lagrangian has the same $T - V$ structure as particle mechanics, but the “kinetic energy” is an integral over continuous fields rather than a sum over discrete masses.

2. The central technical challenge—incorporating vorticity into the Eulerian variational principle—has three solutions: Clebsch variables, Lin constraints, and Euler–Poincaré reduction. All produce the same equations of motion.

3. **Kelvin’s circulation theorem** and **Ertel’s potential vorticity conservation** are Noether consequences of particle-relabeling symmetry in the Lagrangian.

4. **Rotating fluids** are handled by a simple velocity substitution that introduces Coriolis and centrifugal terms. The Coriolis coupling is first-order in velocity—a hallmark of gyroscopic systems.

5. **Superfluids** replace the velocity field with the gradient of a quantum phase, producing irrotational flow with topologically stable quantised vortices.

6. The **two-fluid model** of Khalatnikov describes finite-temperature superfluids as two interpenetrating fluids with a shared Lagrangian, predicting second sound and the fountain effect.

7. The **geometric view** (Arnold, 1966; Arnold and Khesin, 1998; Holm et al., 1998) reveals fluid dynamics as geodesic motion on the diffeomorphism group, connecting it to gauge theory and explaining the onset of turbulence as geodesic instability.
8. The principle of approximating the **Lagrangian first** (Salmon's programme) ensures that approximate models inherit correct conservation laws—a lesson with broad applicability beyond geophysics.

Arnold's central insight was that a fluid is not a collection of particles that happen to be close together, but a single dynamical system whose configuration space is the group of diffeomorphisms of space (Arnold, 1966).

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